numbers 9, 11, 19 which belong to exponent 2 is $\equiv -1 \pmod{20}$. This is a special case of the theorem: *The sum of all the incongruent numbers which belong to exponent 2 with respect to any modulus is* $\equiv -1$. This theorem follows directly from the facts that the order of the product of any two operators of order 2 in G is of order 2 and that -1 corresponds to one of these operators of order 2, since it results from these facts that a change of the signs of all the numbers, except -1, which correspond to operators of order 2 in G, does not affect this totality (mod m).

For the same reason it results that the sum of all the numbers, in any reduced system of residues, which belong to an odd exponent or to twice this exponent is always $\equiv 0$. In fact, if we multiply by -1 all those numbers which belong to any odd exponent or to twice this exponent (mod m) we obtain all those which belong to these exponents. From the theorems stated above, it follows directly that all the sum of all the incongruent numbers which belong to exponent δ mod 2^α is $\equiv 0$ or $\equiv -1$ according as $\delta > 2$ or $= 2$. This follows directly from the fact that $\phi(2^\alpha)$ is $2^{\alpha-1}$, and hence all the exponents to which odd numbers belong (mod 2^α) are powers of 2.

§ 3. Theorems Relating to a Prime Modulus.

We shall first consider the case when $p-1$, p being the prime modulus, is not divisible by the square of a prime number. Hence we have

$$p-1=p_1 p_2 \ldots p_\lambda ,$$

where $p_1, p_2 \ldots p_\lambda$ are distinct prime numbers. Since the sum of the distinct roots of the congruence

$$x^{p_\alpha} \equiv 1 \pmod{p} \qquad \alpha = 1, 2, \ldots \lambda$$

is zero, and unity is one of these roots, it results that the sum of the numbers of the series

$$1, 2, \ldots p-1$$

which belong to exponent p_α is $\equiv -1 \pmod{p}$.

To obtain the sum of those numbers which belong to exponent $p_\alpha p_\beta$ ($\alpha, \beta = 1, 2, \ldots$ or λ and $\alpha \neq \beta$), we observe that the sum of the roots of the congruence

$$x^{p_\alpha p_\beta} \equiv 1 \pmod{p}$$

is zero and that the sum of these roots which belong either to exponent p_α or to exponent p_β is -1. Hence the sum of the roots which belong to exponent $p_\alpha p_\beta$ is 1. These illustrations suffice to suggest the theorem that the sum of the numbers of the set 1, 2, . . . $p-1$ which belong to an exponent which is the product of an even number of distinct primes is $\equiv 1 \pmod p$, while the sum of these numbers is $\equiv -1 \pmod p$ when the exponent is the product of an odd number of distinct prime factors. We proceed to prove, by complete induction, that this theorem is universally true.

Suppose that this theorem is true for r distinct prime factors p_1, p_2, . . . p_r $(r<\lambda)$, and consider the congruence

$$x^{p_1 p_2 \cdots p_{r+1}} \equiv 1 \pmod p.$$

The sums of the roots which belong to a prime exponent and to an exponent which is the product of two, three . . . up to r distinct primes, are, by hypothesis, as follows:

$$-(r+1) + \frac{(r+1)\,r}{2} - \frac{(r+1)r\,(r-1)}{3!} + \cdots + (-1)^{r+1}(r+1).$$

This formula results directly from the fact that a cyclic group has one and only one subgroup whose order is an arbitrary divisor of the order of the group and that the $p_1 p_2 \cdots p_{r+1}$ roots of the congruence

$$p^{p_1 p_2 \cdots p_{r+1}} \equiv 1 \pmod p$$

form a cyclic group (mod p), when they are combined by multiplication.

The terms of the given formula are evidently all the terms, except the first and the last, of the expansion

$$(1-1)^{r+1}.$$

Since the root unity furnishes the first term of this expansion and since the sum of all the roots of the congruence under consideration is $\equiv 0 \pmod p$, it results that the sum of all those roots which belong to exponent $p_1 p_2 \cdots p_{r+1}$ is congruent to the last term of this expansion. That is, this sum is $\equiv 1 \pmod p$ when $r+1$ is even, while it is $\equiv -1 \pmod p$ when $r+1$ is odd. As the given theorem is true when $r=1$ or 2 it must therefore be universally true.

Suppose now that $p-1$ is divisible by p_α^β, p_α being a prime number and $\beta>1$. There will then be one and only one subgroup of order p_α^β in G and the numbers which correspond to the operators of this subgroup are the roots of the congruence

$$x^{p_\alpha^\beta} \equiv 1 \pmod{p}.$$

Since the sums of these roots which satisfy each of the congruences

$$x^{p_\alpha} \equiv 1 \pmod{p}, \quad x^{p_\alpha^2} \equiv 1 \pmod{p}$$

are zero it results directly that the sum of those roots which belong to exponent p_α^2 is zero. If $\beta > 2$, it may be proved in a similar that the sum of those roots which belong to exponent p_α^3 is zero, etc. That is, *the sum of those numbers of the series 1, 2 . . . p—1 which belong to an exponent which is a power, greater than the first, of any prime is* $\equiv 0 \pmod{p}$.

If $p-1$ is divisible by $p_\alpha^\beta q$, q being a prime number, and $\beta > 1$, the sum of those roots of the congruence

$$x^{p_\alpha^\beta q} \equiv 1 \pmod{p},$$

which belong to exponent $p_\alpha^2 q$ is $\equiv 0 \pmod{p}$, since the sum of those whose exponents divide $p_\alpha q$ is $\equiv 0 \pmod{p}$. From this it results directly that the sum of those roots which belong to exponent $p^\delta q$, $\delta \leqq \beta$, is also $\equiv 0 \pmod{p}$, as may be seen by assigning to δ successively the values 3, 4 . . . β. Hence it results from the theorem proved above, by complete induction, that if $p-1$ is divisible by $p_\alpha^\beta q_1$, q_2 . . . q_r, where q_1, q_2 . . . q_r are distinct prime numbers, *the sum of the numbers of the series 1, 2 . . . p−1 which belong to exponent* $p_\alpha^\beta q_1 q_2 \ldots q_r$ *is* $\equiv 0 \pmod{p}$.

We are now in position to prove, by complete induction, that the sum of those numbers of the series 1, 2 . . . p—1 which belongs to an exponent which is divisible by the square of a prime number is always $\equiv 0 \pmod{p}$. In fact we may first prove this for the case when this exponent is of the form $q_1^{a_1} q_2^{a_2}$, q_1 and q_2 being distinct primes, and a_1, $a_2 > 1$. Then we can establish the given result for the case when this exponent is of the form $q_1^{a_1} q_2^{a_2} q_3 q_4 \ldots q_r$ by following the method employed above. After this we establish this result for the exponents which are of the form $q_1^{a_1} q_2^{a_2} q_3^{a_3}$ (a_1, a_2, $a_3 > 1$), and then for the exponents of the form $q_1^{a_1} q_2^{a_2} q_3^{a_3} q_4 \ldots q_r$. As this process can evidently be continued indefinitely, we have established the theorem: *The sum of those numbers of the series 1, 2 . . . p−1 which belong to any exponent which is divisible by the square of a prime number is always* $\equiv 0 \pmod{p}$, *while the sum of those which belong to an exponent which is not divisible by the square of a prime is* $\equiv 1$ *or* $-1 \pmod{p}$ *as the number of the prime numbers which divide this exponent is even or odd.**

*A little more general theorem is given by Bachmann, *Niedere Zahlentheorie* (1902), p. 402.

§ 4. A Few Deductions.

The given theorems may be used to advantage to find the numbers which belong to certain exponents, especially when the ϕ-function of these exponents is small. Since the reciprocal of any number belongs to the same exponent as the number itself, it results that when the ϕ-function of this exponent is 2 the number and its reciprocal constitute the only numbers which belong to this exponent. In particular, when the number n belongs to exponent 3(mod p) n^2 must belong to the same exponent and $n^2+n\equiv -1 \pmod{p}$. Hence we may say that a necessary and sufficient condition that the number $n>1$ belongs to exponent 3 (mod p) is that $n(n+1)\equiv -1$. Hence n belongs to exponent 3 whenever p is of the form $n(n+1)+1$. The five primes below 100 which are of this form are 7, 13, 31, 43 and 73. Numbers belonging to exponent 3 (mod p) may often be readily obtained by the following method, whose correctness is easily proved: Find the reciprocal r of 4 (mod p) and find by trial the smallest value of k such that $kp+r-1$ is a perfect square. The two numbers $\frac{1}{2}(p-1)\pm\sqrt{(kp+r-1)}$ will then belong to exponent 3. It is clear that p must have the form $6n+1$.

In a similar manner we see that if n belongs to exponent 4, n^3 will belong to this exponent and $n^3+n\equiv 0 \pmod{p}$. Two neccessary and sufficient conditions that the number n belongs to exponent 4 (mod p) are therefore that $n(n^2+1)\equiv 0$, and that n is prime to p. Similarly, we observe that a necessary and sufficient condition that n belongs to exponent 6 (mod p) is that $n+\frac{1}{n}\equiv 1 \pmod{p}$. We may deduce from these results the following useful theorem: *A necessary and sufficient condition that the number n, which is prime to the prime odd number p, belongs to exponents 3, 4, or 6 (mod p) is that the sum of n and its reciprocal is congruent to -1, 0, or 1 respectively.* When this sum is congruent to ± 1, n must be prime to p. Hence we have that a necessary and sufficient condition that n belongs to exponent 3 or 6 (mod p) is that $n+\frac{1}{n}\equiv -1$ or $\equiv 1$, respectively.

ON THE REPRESENTATION OF AN INTEGER AS THE SUM OF CONSECUTIVE INTEGERS.

By THOMAS E. MASON, Indiana University.

Lucas has shown that every number not of the form 2^n can be expressed as the sum of two or more consecutive positive integers. In this paper we shall consider series of consecutive integers and shall not exclude zero and negative terms. It is proposed to find the number of ways in

which a number may be expressed as a sum of consecutive integers, including the case of a single term.

The question resolves itself into two problems.

Case 1. *When the number of terms in the series is odd.*

In this case we have

$$(A) \qquad m=(2n+1)a,$$

where m is the number, $2n+1$ the number of terms in the series, and a the mid-term. For every such factorization of m there exists a series with m as the sum, and for every series there exists the corresponding factorization. Hence the problem reduces to the problem of finding the number of ways in which m can be factored into two factors, one of which is odd. When the number is 2^n there is only the one odd factor 1. In every other case if we express m thus,

$$m=2^{a}\cdot p_1^{a_1}\cdot p_2^{a_2}\cdot p_3^{a_3}\cdot p_4^{a_4}\cdot p_5^{a_5}\cdot\cdot\cdot\cdot p_r^{a_r},$$

where the p's are distinct odd primes and the a's are the powers to which they occur, then by the theorem for the number of factors of an integer, we have

$$(a_1+1)(a_2+1)(a_3+1)\cdot\cdot\cdot(a_r+1)$$

as the number of ways in which we can factor m into two factors, one odd.

Case 2. *When the number of terms in the series is even.*

For this case we have

$$(B) \qquad m=2n(2a+1)/2=n(2a+1),$$

where $2n$ is the number of terms and where a and $a+1$ are the pair of terms in the middle of the series. We see that the problem reduces again to that of finding the number of ways in which m can be separated into two factors, one being odd; the result will evidently be the same as in the first case.

Therefore the total number of ways in which m can be expressed as the sum of consecutive integers is

$$2(a_1+1)(a_2+1)(a_3+1)\cdot\cdot\cdot(a_r+1),$$

except in the case where m is 2^n, in which the total number is 2.

Since the factor 2^s has no part in determining the number of ways in which the number may be factored into two factors, one odd, we shall have an infinity of numbers in geometrical series, with ratio two, that can be expressed as the sum of consecutive integers in the same number of ways.

Every factorization of m into two factors, one odd, gives two series. For if $m=(2n+1)a$, we may consider $2n+1$ the number of terms in the series and a the mid-term, or we may take $2a$ as the number of terms and n and $n+1$ as the pair of terms in the middle of the series. Let us consider

$$m=(2n+1)a,$$

where $2n+1$ is the number of terms and a the mid-term. There can be but $a-1$ terms less than a which do not include zero or zero and negative terms. Obviously there will be n terms of the series less than a. Therefore the series will contain all positive terms only when

$$a-n>0. \tag{1}$$

Let us now consider $2a$ the number of terms of the series and n and $n+1$ the pair of terms in the middle of the series. There can be but $n-1$ terms less than n which do not include zero or zero and negatives. Also there can be but $a-1$ terms of the series below n. Therefore the series will contain only positive terms when

$$n-a\geq 0. \tag{2}$$

The relations (1) and (2) are such as to show that when one series contains all positive terms the other will contain a zero or zero and negatives. Since each factorization of m gives one series with all terms positive and one series containing a zero or zero and negatives, we shall have half the total number of series with terms all positive and half with zero or zero and negatives.

From (1) we have that the condition for all positive integers in the series is

$$\text{From (A):}\quad a>n,\quad \frac{m}{2n+1}=a>n,\quad \frac{m}{2n}>\frac{m}{2n+1}>n.$$

$$\therefore \frac{m}{2}>n^2,\ \text{or}\ n<\sqrt{\frac{m}{2}},\ \text{or}\ 2n<\sqrt{(2m)}.$$

That is, when one less than the number of terms is less than the

square root of twice the number (m) the series consists of positive integers. From (B), in like manner we get

$$2a<\sqrt{(2m)}$$

as the condition for all positive terms, where $2a$ is the number of terms in the series.

Since m can be expressed as the sum of n numbers, it can also be expressed as the sum of n series of consecutive integers. The number of such sets of series will equal the combinations of n things, one being taken from each of n groups, where the number in the group is found by the method of this paper. As m can also be expressed as a difference, a product, or a quotient, it can be expressed as the difference, the product, or the quotient, of series or sets of series.

We shall gather our results into the following

THEOREM. *If we consider a number itself as a series of consecutive integers of one term, and do not exclude zero and negative terms, then a number*

$$m=2^{a_1}\cdot p_1^{a_1}.p_2^{a_2}.p_3^{a_3}.p_4^{a_4}\cdot\cdot\cdot p_r^{a_r},$$

where the p's are distinct odd primes and the a's the exponents of the powers to which they appear, can be expressed as the sum of consecutive integers in

$$2(a_1+1)(a_2+1)(a_3+1)\cdot\cdot\cdot(a_r+1)$$

ways, except in the case $m=2^n$ where the number of ways is 2.

One half the number of series will contain an even number of terms and one half an odd number of terms.

Also one half the number of series will be composed of all positive terms and one half will contain zero or zero and negatives.

An example illustrating the above theorem is the following:

$$15=3\times5$$

Therefore the number of series will be

$$2(1+1)(1+1)=2^3=8.$$

Number of Terms.	Mid-Term or Pair of Mid-Terms.	The Series.
1	15	15
3	5	4+5+6
5	3	1+2+3+4+5
15	1	−6−5−4−3−2−1+0+1+2+3+4+5+6+7+8
2	7,8	7+8
6	2,3	0+1+2+3+4+5
10	1,2	−3−2−1+0+1+2+3+4+5+6
30	0,1	−14−13− . . . +0+1 . . . +14+15

NOTE ON PRIME NUMBERS.

By DERRICK N. LEHMER, University of California.

It is a well known theorem that it is possible to find an arbitrarily great number of consecutive composite numbers. This appears from the values which the expression $n!+r$ takes for $r=2, 3, \ldots, n$. This theorem furnishes an interesting proof of the theorem that the number of primes less than or equal to x is not determined by a function of x which is a polynomial in x of finite degree. For if $f(x)$ were such a function of degree n, then for $x=(n+2)!+r$, $f(x)$ must keep the same value for $r=2, 3, 4, \ldots, n+2$. If this value is k, then $f(x)-k=0$ is an equation of degree n with $n+1$ roots, which is impossible.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

362. Proposed by JAMES F. LAWRENCE, Stillwater, Okla.

Show that the number of solutions in positive integers, zero included, of the equation $x+2y+3z=6n$, is $3n^2+3n+1$.

Solution by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Burghfield, England.

$x+2y+3z=6n$. z may have any value from 0 to $2n$, inclusive.

Hence we may assign to it any even or odd value from 0 to $2n$, inclusive.

i. Let $z=2r$. $x+2y=6(n-r)$. y may have any value from 0 to $3(n-r)$.

$\therefore$ There are $3(n-r)+1$ solutions when z is $2r$.

$\therefore$ Total number of solutions for z even is

$$\sum_{r=0}^{r=n}[3(n-r)+1]=(3n+1)(n+1)-\frac{3n.(n+1)}{2}$$

$$=\frac{n+1}{2}(6n+2-3n)=\frac{(n+1)(3n+2)}{2}.$$

ii. Let $z=2r+1$. $x+2y=6(n-r)-3$. y may have any value from 0 to $3(n-r)-2$.

$\therefore 3(n-r)-2+1=3(n-r)-1$ solutions.

$\therefore$ Total number of solutions when z is odd:

$$=\sum_{r=0}^{r=n-1}[3(n-r)-1]=(3n-1)n-\frac{3n(n-1)}{2}=\frac{n}{2}(6n-2-3n+3)=\frac{n}{2}(3n+1).$$

$\therefore$ The total number of solutions

$$=\frac{(n+1)(3n+2)}{2}+\frac{n(3n+1)}{2}=\frac{3n^2+5n+2+3n^2+n}{2}=3n^2+3n+1.$$

Also solved by H. Prime, J. Scheffer, H. C. Feemster, and A. M. Harding.

363. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

(a) If a and n be positive integers, the integral part of $[a+\sqrt{(a^2-1)}]^n$ is odd.

(b) If a and n be positive integers, the integral part of $[\sqrt{(a^2+1)}+a]^n$ is odd when n is even and even when n is odd. [From Todhunter's *Algebra*, p. 353].

I. Solution by the PROPOSER.

Proof. (a) Let $[a+\sqrt{(a^2-1)}]^n=P+Q\sqrt{(a^2-1)}=m$.

Then $[a-\sqrt{(a^2-1)}]^n=P-Q\sqrt{(a^2-1)}=1/[a+\sqrt{(a^2-1)}]^n$.

$\therefore 0<P-Q\sqrt{(a^2-1)}<1$.

Adding m to each member of the inequality $m<2P<m+1$.

Therefore, the integral part of m is odd.

(b) Let $[\sqrt{(a^2+1)}+a]^n=R+S\sqrt{(a^2+1)}=k$.

Then $[-\sqrt{(a^2+1)}+a]^n=R-S\sqrt{(a^2+1)}=\left(\frac{-1}{\sqrt{(a^2+1)}+a}\right)^n$.

If n is even, $0<R-S\sqrt{(a^2+1)}<1$. Adding k, $k<2R<k+1$. Whence the integral part of k is odd.

If n is odd, $-1<R-S\sqrt{(a^2+1)}<0$. Adding k, $k-1<2R<k$. Whence the integral part of k is even.

Also solved similarly by S. Lefschetz.

II. Solution by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Burghfield, England, and J. SCHEFFER, A. M., Hagerstown, Maryland.

Let I be the integral part of $[a+\sqrt{(a^2-1)}]^n$.

Then $I+F=[a+\sqrt{(a^2-1)}]^n$, where F is a proper fraction,

$$=a^n+na^{n-1}\sqrt{(a^2-1)}+\frac{n.(n-1)}{2!}(a^2-1)+\frac{n.(n-1)(n-2)}{3!}(a^2-1)^{\frac{3}{2}}+\ldots$$

Also, $a>\sqrt{(a^2-1)}$; $\therefore a-\sqrt{(a^2-1)}<1$, and $[a-\sqrt{(a^2-1)}]^n<1$.

Let $F'=[a-\sqrt{(a^2-1)}]^n=a^n-na^{n-1}\sqrt{(a^2-1)}$

$$+\frac{n.(n-1)}{2!}(a^2-1)-\frac{n.(n-1)(n-2)}{3!}(a^2-1)^{\frac{3}{2}}+\ldots$$

$$\therefore I+F+F'=2\left(a^n+\frac{n.(n-1)}{2!}(a^2-1)+\ldots\right)=2p \text{ (say)}.$$

Hence, $F+F'=1$, and $I=2p-1$ and is odd.

Similarly, $I+F=[\sqrt{(a^2+1)}+a]^n=(a^2+1)^{\frac{1}{2}n}+n(a^2+1)^{\frac{1}{2}(n-1)}$

$$+\frac{n.(n-1)}{2!}(a^2+1)^{\frac{1}{2}(n-1)}a^2+\ldots$$

$$F'=[\sqrt{(a^2+1)}-a]^n=(a^2+1)^{\frac{1}{2}n}-n(a^2+1)^{\frac{1}{2}(n-1)}a+\ldots$$

(i) n odd. $I+F-F'=2[n(a^2+1)^{\frac{1}{2}(n-1)}a+\ldots]=2p$ (say)=an even integer.

$\therefore F-F'=0$, and I is an even integer.

(ii) n even. $I+F+F'=2[(a^2+1)^{\frac{1}{2}n}+\frac{n.(n-1)}{2!}(a^2+1)^{\frac{1}{2}(n-2)}a^2+\ldots]$=an even integer.

$\therefore F+F'=1$, and I is an odd integer.

GEOMETRY.

389. Proposed by H. PRIME, Boston, Mass.

On the same side of a given base, triangles are erected such that the bisectors of their vertex angles all pass through a given point. Find the locus of the vertices (i) when the vertex angle are all equal, (ii) when the vertex angles are all unequal.

Solution by the PROPOSER.

Take the given base $BB=2m$ for the axis of x, its midpoint O for the origin. Let the triangles be erected on the positive side of BB; a, b be the given point, P any point of the required locus, and the circle BPB, radius r, cut OY in Q. The coördinates of Q are $0, \pm\sqrt{(r^2-m^2)}-r$. The angle bisector passes through Q and a, b. Its equation is

$$y=[b\mp\sqrt{(r^2-m^2)}+r]x/a\pm\sqrt{(r^2-m^2)}-r \ldots (1).$$

The equation of the circle BQB is

$$x^2+[y\mp\sqrt{(r^2-m^2)}]=r^2 \ldots (2).$$

The upper sign apply when the center of the circle is above the axis of x, the lower sign when it is below.

Eliminating the variable parameter r between (1) and (2), we have

$$(A)\quad 2ay^2+2bxy+(a-x)(x^2+y^2-m^2\mp\sqrt{[(x^2+y^2-m^2)^2+4m^2y^2]},$$
$$\text{or, } y(ay-bx)^2-(a-x)(x^2+y^2-m^2)(ay-bx)=m^2y(a-x)^2.$$

This is the general equation of the locus and defines its form in every relation of the constants. If $a=0$, (A) reduces to either

$$(B)\qquad x=0, \text{ or,} \qquad (C)\qquad x^2+y^2-y(b^2-m^2)/b=m^2.$$

(B) applies when the triangles are isosceles, their vertex angles unequal and bisected by the axis of y.

(C) is the equation of a circle whose radius$=(b^2+m^2)/2b=$a constant. Hence all the triangles are inscribed in the same circle and, having the same base, their vertex angles are equal.

Also solved by C. N. Schmall.

390. Proposed by PROF. R. C. ARCHIBALD, Brown University, Providence, R. I.

Find, geometrically and without introducing focal properties, the locus of the vertices of the conjugate parallelograms of an ellipse.

Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

Let $y=mx+\sqrt{(a^2m^2+b^2)}$, $y=m'x+\sqrt{(a^2m'^2+b^2)}$ be the equation of the tangents parallel to two conjugate diameters. But between m and m' there is the well known relation: $mm'=-(b^2/a^2)$, whence $m'=-(b^2/a^2m)$. Substituting this in the second of the above equations, we have $a^2ym+b^2x=ab\sqrt{(a^2m^2+b^2)}$, and dividing the last equation by $y-mx=ab\sqrt{(a^2m^2+b^2)}$, we obtain

$$m=\frac{b}{a}\cdot\frac{ay-bx}{ax+by};$$

and substituting in the latter equation, we finally get $a^2y^2+b^2x^2=2a^2b^2$, the equation of a concentric ellipse with the semi-axes $a\sqrt{2}$ and $b\sqrt{2}$.

391. Proposed by W. J. GREENSTREET, M. A., Editor, The Mathematical Gazette, Burghfield, England.

An ellipse is inscribed in the triangle of reference and has one focus at (sec$A$, sec$B$, sec$C$). Find the other focus and the sum of the squares of the axes of the ellipse.

Solution by WILLIAM HOOVER, Ph. D., Athens, Ohio.

In trilinear coördinates, let $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ be the two foci. Then the first is found by

$$\alpha_1\cos A=\beta_1\cos B=\gamma_1\cos C\ldots(1),$$
$$\text{and } a\alpha_1+b\beta_1+c\gamma_1=2\triangle\ldots(2),$$

$$\text{or, } \alpha_1=2R\cos B\cos C,\ \beta_1=2R\cos A\cos C,\ \text{and } \gamma_1=2R\cos A\cos B\ldots(3),$$

R being the radius of the circum-circle.

If b_1 be the semi-minor axis of the ellipse,

$$\alpha_1\alpha_2=b_1^2=\beta_1\beta_2=\gamma_1\gamma_2\ldots(4),\ \text{or,}$$
$$\cos B\cos C.\alpha_2=\cos A\cos C.\beta_2=\cos A\cos B.\gamma_2\ldots(5),$$

which with (3) gives

$$\alpha_2=R\cos A,\ \beta_2=R\cos B,\ \gamma_2=R\cos C\ldots(6).$$

(4) now gives $b_1^2=2R^2\cos A\cos B\cos C\ldots(7)$.

Also, d=the distance between the ortho-center and circum-center, is given by

$$d^2=R^2(1-8\cos A\cos B\cos C)\ldots(8).$$

If $2a_1$ be the major axis of the ellipse, $a_1^2=\frac{1}{4}d^2+b_1^2=\frac{1}{4}R^2$, or $2a_1=R$, and determining $4a_1^2+4b_1^2$.

392. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

A tangent to a curve at any point P cuts the tangent and the normal at a fixed point O in the points M and N, and the rectangle $OMP'N$ is completed. Find the curve which is such that the triangle formed by the tangents at any three points P, Q, R is equal to the triangle formed by the corresponding points P', Q', R'.

No solution of this problem has been received.

CALCULUS.

316. Proposed by C. N. SCHMALL, New York City.

$$\int_0^\infty \frac{\cos ax}{1+x^2}dx=\tfrac{1}{2}\pi e^{-a}=\int_0^\infty \frac{x\sin ax}{1+x^2}dx.$$

(From Bromwich, *Theory of Infinite Series*, p. 442, ex. 5, and also from Carslaw, *Fourier's Series*, p. 113, ex. 12.) Prove this by any method.

II. Solution by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Burghfield, England.

Lemma. $u=\underset{h=0}{L}[\sin h+\frac{1}{2}\sin 2h+\frac{1}{3}\sin 3h+\ldots]=\underset{h=0}{L}\frac{\pi-h}{2}$, the sum being taken between 2π and small values of h, $=\frac{1}{2}\pi$.

$\therefore u=\int_0^\infty \frac{\sin x}{x}dx=\frac{1}{2}\pi$, and it is also clear that $\int_0^\infty \frac{\sin ax}{x}dx=\frac{1}{2}\pi$.

Now let $U=\int_0^\infty \frac{x\sin ax\,dx}{1+x^2}$; $U-\frac{1}{2}\pi=\int_0^\infty \frac{x\sin ax\,dx}{1+x^2}-\int_0^\infty \frac{\sin ax}{x}$

$$=-\int_0^\infty \frac{\sin ax}{x(1+x^2)}dx\ldots(1).$$

Differentiating twice with respect to a,

$$\frac{d^2U}{da^2}=\int_0^\infty \frac{x\sin ax}{1+x^2}dx=U.$$

Multiplying by $\frac{dU}{da}$, $\frac{d^2U}{da^2}\cdot\frac{dU}{da}=\frac{U.dU}{da}$. $\frac{1}{2}\frac{d}{da}\left(\frac{dU^2}{da^2}\right)=\frac{1}{2}\frac{d(U^2)}{da}$, $\left(\frac{dU}{da}\right)^2=U^2+\kappa$,

$$\frac{dU}{\sqrt{(U_2+\kappa)}}=da.$$

$\therefore \log[U+\sqrt{(U^2+\kappa)}]=a+\lambda,\ U+\sqrt{(U^2+\kappa)}=e^{a+\lambda}.$

Also, $\sqrt{(U^2+\kappa)}-U=\kappa e^{-(a+\lambda)},\ 2U=e^{a+\lambda}-\kappa e^{-(a+\lambda)}=Ce^{-a}+C'e^{a}$, where C, C', are constants. U not increasing indefinitely with a it follows that $C'=0$. When a is very small, (1) becomes

$$\lim_{a=0} U-\tfrac{1}{2}\pi=\lim_{a=0}-\int_0^\infty \frac{adx}{1+x^2}=\lim_{a=0}-\frac{a\pi}{2}=0;\ \therefore C=\tfrac{1}{2}\pi, \text{ and}$$

$$u=-\int_0^\infty \frac{x\sin axdx}{1+x^2}=\frac{\pi}{2}e^{-a}\ (a \text{ being positive}).$$

But $\int_0^\infty \frac{x\sin ax}{1+x^2}dx=\frac{\pi}{2}-u=\frac{\pi}{2}(1-e^{-a}).$

Differentiating with respect to a,

$$\int_0^\infty \frac{x\cos ax}{1+x^2}dx=\frac{\pi}{2}e^{-a}.$$

$\therefore$ etc. (Cf. Roberts' *Treatise on the Integral Calculus*, Part I, p. 181.)

317. Proposed by C. N. SCHMALL, New York City.

A generating line of a right circular cylinder passes through the center of a sphere. The diameter of the cylinder is less than the radius of the sphere. Show that the surface of the cylinder included within the sphere is given by an elliptic integral.

Solution by A. M. HARDING, Fayetteville, Arkansas.

Let a=diameter of cylinder; r=radius of sphere. Choose the generating line of cylinder for z-axis. Let equation of sphere and cylinder be

$$x^2+y^2+z^2=r^2 \text{ and } x^2+y^2=ax,$$

respectively. Then

$$\frac{A}{4}=\int\int\left[1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2\right]^{\frac{1}{2}}dzdx.$$

Eliminate y and obtain $z^2+ax=r^2$. Hence z-limits are 0 and $\sqrt{(r^2-ax)}$, x-limits are 0 and a.

From equation of cylinder, we find

$$\frac{\partial y}{\partial x}=\frac{a-2x}{2y},\quad \frac{\partial y}{\partial z}=0.$$

$$\therefore \frac{A}{4}=\int_0^a \int_0^{\sqrt{(r^2-ax)}}\left[1+\left(\frac{a-2x}{2y}\right)^2\right]^{\frac{1}{2}}dzdx=\int_0^a \int_0^{\sqrt{(r^2-ax)}}\frac{a}{2\sqrt{(ax-x^2)}}dzdx,$$

since $y^2=ax-x^2$.

$$\therefore A=2a\int_0^a \sqrt{\left(\frac{r^2-ax}{ax-x^2}\right)}dx. \quad \text{Putting } x=a\sin^2\phi,$$

$$A=2a\int_0^{\frac{1}{2}\pi}\frac{\sqrt{(r^2-a^2\sin^2\phi)}}{\sqrt{(a^3\sin^2\phi-a^3\sin^4\phi)}}.2a\sin\phi\cos\phi\,d\phi$$

$$=4a\int_0^{\frac{1}{2}\pi}\sqrt{(r^2-a^2\sin^2\phi)}.\frac{a\sin\phi\cos\phi}{a\sin\phi\sqrt{(1-\sin^2\phi)}}d\phi$$

$$=4ar\int_0^{\frac{1}{2}\pi}\sqrt{\left(1-\frac{a^2}{r^2}\sin^2\phi\right)}d\phi=4ar\,E\left(\frac{a}{r},\frac{\pi}{2}\right),\ a<r.$$

Also solved by Francis Rust and J. Scheffer.

318. Proposed by JOHN C. GREGG, Greencastle, Ind.

A thread is wound spirally n times around a cone, the radius of whose base is r, and slant height h, the turns being at uniform distance apart. If the thread is kept taut, what will be the length of the trace of its end on a horizontal plane?

No correct solution of this problem has been received. Professor Feemster has given a solution finding the length of the thread. But the problem does not require that. Let us have a number of solutions of this problem. ED. F.

319. Proposed by C. N. SCHMALL, New York City.

Given $u=\frac{yz}{x}$, $v=\frac{zx}{y}$, $w=\frac{xy}{z}$, prove

$$\begin{vmatrix} \frac{\partial u}{\partial x}, & \frac{\partial u}{\partial y}, & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x}, & \frac{\partial v}{\partial y}, & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x}, & \frac{\partial w}{\partial y}, & \frac{\partial w}{\partial z} \end{vmatrix}=4xyz.$$

Solution by S. LEFSCHETZ, Ph. D., The University of Nebraska.

If Δ is the determinant proposed, we have:

$$\Delta=\begin{vmatrix} -\frac{yz}{x^2}, & \frac{z}{x}, & \frac{y}{x} \\ \frac{z}{y}, & -\frac{xz}{y^2}, & \frac{x}{y} \\ \frac{y}{z}, & \frac{x}{z}, & -\frac{xy}{z^2} \end{vmatrix} = xyz \begin{vmatrix} -\frac{1}{x}, & \frac{1}{y}, & \frac{1}{z} \\ \frac{1}{x}, & -\frac{1}{y}, & \frac{1}{z} \\ \frac{1}{x}, & \frac{1}{y}, & -\frac{1}{z} \end{vmatrix} = xyz \begin{vmatrix} 0, & 0, & \frac{2}{z} \\ \frac{2}{x}, & 0, & 0 \\ \frac{1}{x}, & \frac{1}{y}, & -\frac{1}{z} \end{vmatrix} = 4.$$

The value announced, $4xyz$, is therefore erroneous.

Solved similarly by J. Scheffer and W. J. Greenstreet.

MECHANICS.

356. Proposed by the late G. B. M. ZERR, Ph. D.

A cantilever beam length a is loaded with c pounds per running foot at its fixed end and increases uniformly to b pounds per running foot at its free end. Find the deflection at the free end due to this load.

Solution by FRANCIS RUST, E. E., Pittsburg, Pa.

The problem is solved by determining the elastic curve of the beam's axis from its second differential equation

$$y''=\frac{M}{IE},$$

as derived from Hooke's law, to be found in all text books. M is the bending moment in the point, abscissa$=x$; I is the moment of inertia of the beam's cross-section, and E is the modulus of elasticity of its material.

M is to be determined by the arrangement of the load with mathematical certainty. Taking the point, abscissa$=x$, for origin,

$$M=\int_0^{a-x} q\,\xi\,d\,\xi,$$

q, the load per lineal foot, to be expressed as a function of the variable of integration ξ. $q=c+px$, and $c+pa=b$. Consequently,

$$p=\frac{b-c}{a},$$

and q as a function of ξ is

$$q=c+\frac{b-c}{a}.(x+\xi)=\frac{ac+(b-c)x}{a}+\frac{b-c}{a}\xi, \text{ and}$$

$$M=\frac{ac+(b-c)x}{a}\int_0^{a-x}\xi\,d\xi+\frac{b-c}{a}\int_0^{a-x}\xi^2 d\xi=\frac{ac+(b-c)x}{a}.\frac{(a-x)^2}{2}$$

$$+\frac{b-c}{a}.\frac{(a-x)^3}{3}=\frac{a^2}{6}\left(1-\frac{x}{a}\right)^2\left((2b+c)+(b-c)\frac{x}{a}\right)$$

which, properly simplified, gives us

$$M=\frac{a^2}{6}\left[(2b+c)-3(b+c)\frac{x}{a}+3c\left(\frac{x}{a}\right)^2+(b-c)\left(\frac{x}{a}\right)^3\right].$$

Assuming now a beam of uniform cross-section and homogeneous material, and introducing the new variable, $x/a=u$, we have $dx=adu$.

The integration of $y''=M/IE$ may be performed without any difficulty. It gives the equation of the elastic curve,

$$y=\frac{a^4}{6EI}.\left[\frac{2b+c}{2}\left(\frac{x}{a}\right)^2-\frac{b+c}{2}\left(\frac{x}{a}\right)^3+\frac{c}{4}\left(\frac{x}{a}\right)^4+\frac{b+c}{20}\left(\frac{x}{a}\right)^5\right]$$

from which is derived for $x=a$,

$$y_a=f=\frac{a^4(11b+4c)}{120EI}.$$

Remark. This result is as simple as it is wrong.

In my memoir *Der Fehler in Hooke's Gesetz*, published in Oesten, Wochensch. f. d. aff. Bandienst, year 1900, p. 252, ff., I have deduced mathematically and proved by experiment, that Hooke's law *cannot be applied on the beginning of deformation.*

The locus of any elastic curve, deduced from Hooke's law, *p. e.* our equation above, is a continuous curve, whilst the experiment shows, that a cantilever (spring under its own weight, for instance) will remain *unbent for a certain length at its free end.*

The investigation into these conditions reveals the fact, that any material may be submitted to a certain stress, which must be overcome before deformation takes place. Such minimal stress may be called *the resistance of inertia* (Tragheits widerstand). Denoting it by a, the true expression of the law of elastic deformation is

$$\triangle l:l=\sqrt{(\sigma^2+a^2)}:E.$$

PROBLEMS FOR SOLUTION.

ALGEBRA.

369. Proposed by WILLIAM HOOVER, Ph. D., Athens, Ohio.

If $f(m)=(1+x)^m$, and $f(n)=(1+x)^n$, why not obviously $f(m).f(n)=(1+x)^{m+n}=f(m+n)$?

370. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Prove that, if the fraction m/p (p prime) gives a recurring decimal with an even number of digits in the cycle, the sum of the two halves will be composed of 9's. (The special case where the number $=p-1$ was proposed in the MONTHLY, Vol. IV, and answered in Vol. V, p. 11. The proof there given, however, is not complete.) The above property is true of other fractions, *e. g.*, $\frac{1}{77}=.\dot{0}12987\dot{}$, $\frac{1}{133}=\frac{1}{7.19}=.\dot{0}07518796,992481203\dot{}$. Find for what fractions this is true.

371. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

In a G. P. of an odd number of terms, all of the terms being positive, and the ratio different from 1, show that the middle term is less than the arithmetical mean.

372. Proposed by S. LEFSCHETZ, Ph. D., University of Nebraska.

Prove that $\sum_{n=1}^{n=\infty} n^2 x^{n-1}=\frac{1+x}{(1-x)^3}$, if mod.$x<1$. (*Schlömilch.*)

373. Proposed by X, National Electric Light Association, Brooklyn, N. Y.

(a) For underground distribution of direct current electrical energy, we have $DA^n-CB^n=H$, where the only unknown is n, which represents the number of years it will take a large direct current low tension feeder to pay by line loss saving for the increased investment over a smaller feeder.

(b) $bVf^2l^2\beta^2+aVf\beta^{1.6}=W$, the iron loss equation which is to be solved for β. When $bVf^2l^2\beta^2$ represents the eddy current loss in the core of a transformer, and $aVf\beta^{1.6}$ is the hysteresis loss in the core. a, b, and l are constants of the core, V is the voltage, f is the frequency, and β is the flux density and is the only unknown in the equation.

Letting $bVf^2l^2=A$, and $aVf=C$, we have $A\beta^2+C\beta^{1.6}=W$.

GEOMETRY.

401. Proposed by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

Find by Euclidean geometry a point whose distances from the vertices of an equilateral triangle are in the ratio 3:4:5. The general case of ratio $a:b:c$ would prove interesting.

402. Proposed by H. PRIME, Boston, Mass.

The diameter of a hoop-shaped ring (or collar) is 24 inches at one edge and 28 inches at the other edge. A cross-section is a crescent with circular arcs of 120° and 60°, whose common chord is 4 inches long. Find its volume by elementary methods (without the use of calculus or the center of gravity).

CALCULUS.

322. Proposed by E. B. ESCOTT, University of Michigan.

Find the equation of the curve such that the solid of revolution generated by revolving it about the x-axis shall have a volume equal to the m/nth part of the volume of the circumscribed cylinder.

323. Proposed by C. N. SCHMALL, New York City.

From what height must an elastic ball be dropped in order that, after impact with the hard surface of the sidewalk, it may rebound to a given altitude a in the least possible time from the moment of descent?

324. Proposed by V. M. SPUNAR, Chicago, Ill.

A parabola slides between two rectangular axes; find (a) the locus of the focus, and (b) the locus of the vertex.

NOTES AND NEWS.

In the May, 1911, number of the *Mathematical Gazette*, page 93, there occurs the following statements: "If modern American text-books in arithmetic offer any criterion of the teaching of the subject in the States, it would appear that America lags behind this country (England) and, indeed, most European countries. For the most part they are characterized by a drab monotony of treatment, by a lack of vitality, and by an entire absence of any correlation with other branches of a mathematical education." M.

Macmillan and Company recently issued volume II of the *Theory of Determinants* in the historical order of development by Dr. Thomas Muir, Superintendent of Education in Cape Colony, Africa. This volume covers the period from 1841 to 1860 and is very similar to volume I in general arrangement. It shares also with volume I the decided defect that the articles quoted are not always assigned to the year of publication but they frequently bear an earlier date, in case such a date appears at the end of the article. In a historical work this arrangement is very confusing since it is not in accord with the present common practice. For instance, if one reads, on page 81 of this book, that the second edition of the "Elementary theorems relating to the determinants, by Spottiswoode, appeared in Crelle's Journal in 1853, and then turns to page 46, volume I, of the *Encyklopaedie der Mathematischen Wissenschaften* and finds the date 1856 assigned to the same article, one is naturally annoyed. Some of the most careful works of reference, like the *Encyclopédie des Sciences Mathématiques*, give both the date of publication and also the date when the article is supposed to have been completed, in case the latter is known. When only one date is given it should always be the former, according to modern practice. M.

BOOKS.

A History of the Theories of Aether and Electricity From the age of Descartes to the close of the nineteenth century. By E. T. Whittaker, Hon. Sc. D. (Dubl.); F. R. S.; Royal Astronomer of Ireland. 8vo. Cloth, xiii+475 pages. Price, $4.50. New York and London: Longmans, Green, & Co.

The author has given in this book a very complete history of the various theories respecting the luminiferous ether held by all the great thinkers since the time of Descartes. In most instances where mathematical treatment is necessary the author uses Vector Analysis, thus making some parts difficult to read for those who are unfamiliar with that branch of mathematics. An idea of the scope of the work may be obtained from the table of contents. The various subjects come under twelve chapters, as follows: Chapter I, The Theory of the Aether in the 17th Century; Chapter II, Electric and Magnetic Science, prior to the Introduction of Potentials; Chapter III, Galvanism, from Galvani to Ohm; Chapter IV, The Luminiferous Medium, from Bradley to Fresnel; Chapter V, the Aether as an Elastic Solid; Chapter VI, Faraday; Chapter VII, The Mathematical Electricians of the Middle of the Nineteenth Century; Chapter VIII, Maxwell; Chapter IX, Models of the Aether; Chapter X, The Followers of Maxwell; Chapter XI, Conduction in Solutions and Gases, from Faraday to J. J. Thomson; Chapter XII, The Theory of Aether and Electrons in the Closing Years of the 19th Century.

Every teacher of Physics will want this thoroughly scholarly work in his library. F.

Advanced Calculus. By Edwin Bidwell Wilson, Associate Professor of Mathematics, Massachusetts Institute of Technology. 8vo. Cloth, ix+566 pages. Price, $5.00. Boston and Chicago: Ginn & Co.

"Professor Wilson's Advanced Calculus supplies in a single volume a comprehensive second course in calculus. Although modern rigorous tendencies are given due attention, the chief aim of the book is to confirm and to extend the student's knowledge of the great formal methods of analyais that are essential alike to the practical and to the pure mathematician. To connect with elementary texts, two chapters in review are supplied, and many subsequent chapters are tempered with material which is essentially review. Advanced differential calculus is represented by work on Taylor's formula, with special reference to approximate analysis, partial differentiation of explicit nnd implicit functions, complex numbers, and vectors. As an extension of previous formal work in integration, four chapters are given to the integration of differential equations. In integral calculus line integrals, multiple integrals, infinite integrals, special functions defined by integrals, and the calculus of variations are treated. Then follow chapters on series, special develments, functions of a complex variable, elliptic functions, and functions of real variables. Throughout the work especial attention has been paid to the needs of students of applied mathematics and mathematical physics. A very large number of exercises have been provided, and every attempt has been made to furnish a thorough, practical, teachable, live textbook and laboratory manual of higher calculus."

The publishers of this work have again put teachers of the Calculus under great obligation to them by bringing out this splendid treatise which cannot be of great advantage commercially, but which will be of immense value pedagogically. They, as well as its author, deserve the thanks of American mathematicians. F.

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THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XIX. APRIL, 1912. NO. 4.

SOME USEFUL MATHEMATICAL BOOKS BEYOND ELEMENTARY CALCULUS.

By DR. G. A. MILLER, University of Illinois.

There is probably always a considerable number of young men who would like to extend their knowledge of mathematics but who are in doubt as to the most useful books. This applies especially to those who do not have easy access to good libraries and who have never enjoyed good educational advantages. The following suggestions are intended mainly for this class, but it is hoped that some others may be able to derive profit from reading them. The fact that there are so many good recent American mathematical books is especially encouraging, and one of our objects is to call attention to this important fact.

We first inquire, what are the most useful works along the line of algebra for a student who has had only an elementary course in this subject. Doubtless all would agree that such a student should secure Bôcher's *Introduction to Higher Algebra*, published by the Macmillan Company in 1907. He may find Bôcher's treatment of many subjects limited and too concise, and it is hoped that he will be led to look elsewhere for greater details as well as for algebraic subjects which he does not find in this work; but he will probably appreciate the methods and the subject matter of this work more and more as he advances.

In particular, he may desire to learn something about the fundamental properties of numbers. In this case he would probably find Reid's *Theory of Algebraic Numbers*, published by the Macmillan Company in 1910, very inspiring as well as very elementary. Cajori's *Theory of Equations*, published by the same company in 1904, would serve very well to secure a deeper knowledge of the fundamental subject of equations, and this knowledge could be extended along important lines by reading Dickson's *Theory of Algebraic Equations*, published by Wiley and Sons in 1903. Among tne classic algebras in foreign languages the following are especially worth having: Weber's *Lehrbuch der Algebra*, second edition, published at Braunschweig, Germany, by Vieweg und Sohn, 1898-1908, three volumes of 703, 855, and